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All Variations on Perfectly Orderable Graphs

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An ordered graph is a graph whose vertices are positive integers. Two ordered graphs are isomorphic if the order-preserving bijection between their sets of vertices is a graph isomorphism. We identify the family of all sets S of ordered graphs with the following properties: (1) Each member of S is a P_4 (defined as a chordless path with four vertices and three edges). (2) If an ordered graph Z has no induced subgraph isomorphic (as an ordered graph) to a member of S, then Z is perfect. This work is related to Berge's Strong Perfect Graph Conjecture and was motivated by Chvátal's theorem on perfectly orderable graphs.

1. THE RESULTS

Claude Berge defined a graph G to be *perfect* if for every induced subgraph H of G, the chromatic number $\chi(H)$ of H equals the largest number $\omega(H)$ of vertices in a clique of H.

Chvátal [1] introduced a class of perfect graphs, called *perfectly orderable graphs* and characterized by the existence of a linear order $<$ on the set of vertices such that no chordless path with vertices a, b, c, d and edges ab, bc, cd has $a<b$ and $d<c$.

This paper reports the results of a search for all variations on perfectly orderable graphs in a sense that we are about to make precise.

By an *ordered graph* we shall mean a graph whose vertices are distinct positive integers; two ordered graphs $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ will be called *isomorphic* if there exists a bijection $f: V_1 \to V_2$ such that

$$f(i)<f(j) \qquad \text{if and only if} \quad i<j$$

and such that

$$f(i)f(j)\in E_2 \qquad \text{if and only if} \qquad ij\in E_1.$$

By the *trace* of an ordered graph Z, we shall mean the subset S of the set of the 12 ordered graphs in Fig. 1 such that $X\in S$ if and only if Z contains

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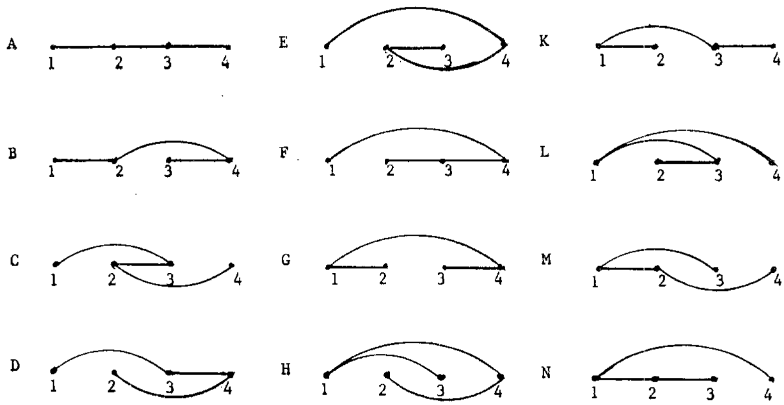

FIGURE 1

an induced ordered subgraph isomorphic to X. To simplify notation, we shall omit the braces when describing subsets of $\{A, B, ..., N\}$: thus we shall write BDF for $\{B, D, F\}$ and so on.

Now Chvátal's theorem may be restated as follows:

> if the trace of an ordered graph Z is disjoint from BDF then Z is perfect.

Our aim is to identify *all* sets S that could replace BDF in this theorem. The main results of this work involve the following sets:

$S_1 = ACEFK$	$S_2 = ACEMN$	$S_3 = CEGK$
$S_4 = ABHL$	$S_5 = ACEKN$	$S_6 = ABCEGH$
$S_7 = ACEKM$	$S_8 = AEGKM$	$S_9 = CDFG$
$S_{10} = ACDEK$	$S_{11} = ADEGK$	$S_{12} = ABCLN$
$S_{13} = ACDFL$	$S_{14} = BCGL$	$S_{15} = AEHK$
$S_{16} = ABCFL$	$S_{17} = ACGHKL$	$S_{18} = ABCDL$
$S_{19} = ABDGL$	$S_{20} = CGMN$	$S_{21} = ABCLM$
$S_{22} = ABGLM$	$S_{23} = EGHKM$	$S_{24} = DFGHK$
$S_{25} = DEGHK$	$S_{26} = EFGHK$	$S_{27} = CEFHK$
$S_{28} = EGHKN$	$S_{29} = CEHKN$	$S_{30} = BDGHL$
$S_{31} = BGHMN$	$S_{32} = BGHLM$	$S_{33} = BGHLN$
$S_{34} = BCHLN$	$S_{35} = BFGHL$	$S_{36} = BCFHL$
$S_{37} = BDF$	$S_{38} = KMN$	$S_{39} = LMN$
$S_{40} = DEF$	$S_{41} = ABCGHK$	$S_{42} = ACEGHL.$

THEOREM 1. *If the trace of an ordered graph Z is disjoint from some S_i with $i = 1, 2, ..., 42$ then Z is perfect.*

(To anticipate, the statements for $i = 1, 2, ..., 36$ are a consequence of Hayward's theorem [3] that weakly triangulated graphs are perfect; the statements for $i = 37, 38, 39, 40$ are equivalent formulations of Chvátal's theorem [1]. Finally, the statements for $i = 41, 42$ follow from the perfection of opposition graphs which we prove in Theorem 2.)

Given a graph $G = (V, E)$ and a linear order $<$ on the set of its vertices, we shall write $u \to v$ if and only if $uv \in E$ and $u < v$. *A* P_4 (defined to be a chordless path with four vertices) involving vertices u, v, w, z and edges uv, vw, wz will be said to be *bad* if $u \to v$ and $w \to z$. As usual we shall let $N(u)$ stand for the set of neighbours of u in G; $\bar{G}$ will denote the complement of G.

A part of our argument used in the proof of Theorem 1 is of an independent interest; we shall set it on its own as Theorem 2. This theorem involves the notion of *opposition graph*: this is a graph with a linear order $<$ on the set of its vertices such that no induced P_4 is bad.

A *star-cutset* in a graph G is a nonempty set of vertices S such that some vertex $w \in S$ is adjacent to all remaining vertices in S. We shall say that the star-cutset is *centered* at w.

A graph G is said to be *minimal imperfect* if G is not perfect, but every proper induced subgraph H of G is perfect. We shall rely on the following result (see Chvátal [2]):

THE STAR-CUTSET LEMMA. *No minimal imperfect graph has a star-cutset.*

THEOREM 2. *An opposition graph G is either bipartite or else its complement $\bar{G}$ has a star-cutset.*

Our two theorems will be proved in reverse order.

2. PROOFS

LEMMA 1. *If in an opposition graph $G = (V, E)$ there exist vertices a, c such that*

(i) $ac \notin E$, $N(a) \cap N(c) \neq \varnothing$,

(ii) $c \to x$ *whenever* $x \in N(a) \cap N(c)$,

(iii) $a \to y$ *whenever* $y \in N(a) - N(c)$,

then $\bar{G}$ has a star-cutset.

Proof of Lemma 1. Note first that if $N(a) - N(c) = \varnothing$ then trivially

$V-(\{c\} \cup N(c))$ is a star-cutset in $\bar{G}$ centered at a. If $N(a)-N(c) \neq \emptyset$ then note that for every vertex u in $N(a)-N(c)$ and any vertex v in $N(a) \cap N(c)$ we must have $uv \in E$, for otherwise $cvau$ would be a bad P_4. But now $V-N(a)$ is a star-cutset in $\bar{G}$ centered at a. ▮

A vertex s will be said to be a *source* if $w \to s$ for no $w \in V$; a vertex s will be said to be a *sink* if $s \to w$ for no $w \in V$; a vertex will be said to be *mixed* if it is neither a source nor a sink.

Proof of Theorem 2. Assume the theorem true for graphs with fewer vertices than G. If the graph is bipartite then there is nothing to prove, else there must exist a nonbipartite component G' of G. If this component has fewer vertices than G, we are done by the induction hypothesis: every star-cutset of $\bar{G}'$ extends into a star-cutset of $\bar{G}$. Thus we may assume that G is connected and not bipartite.

Now the following algorithm finds a star-cutset in $\bar{G}$.

Let $S=\{a_1, a_2, ..., a_k\}$ be the set of sources.

Step 1. If there are two sources a_i, a_j $(i \neq j)$ and a vertex u such that $a_i \to u$, $a_j \to u$ then Lemma 1 applies with $a=a_i$, $c=a_j$, and so $\bar{G}$ has a star-cutset.

Step 2. Write $w \in P_i$ iff $a_i \to w$ for $i=1, 2, ..., k$. Since we did not stop in Step 1, $P_i \cap P_j=\emptyset$ for all $i \neq j$. Write

$$P=\bigcup_{i=1}^{k} P_i.$$

If $uv \in E$ for some u in P_i, v in P_j with $i \neq j$ then stop: $\bar{G}$ has a star-cutset. [To prove this claim, find the first vertex u in the linear order $<$ for which there are an edge $uv \in E$ and different subscripts i, j such that $u \in P_i$, $v \in P_j$. Now apply Lemma 1 with $c=u$, $a=a_j$.]

Step 3. Write $Q=V-(S \cup P)$. If there exists a vertex $w \in P_j$ such that $N(w) \cap Q=\emptyset$ then stop: $\bar{G}$ has a star-cutset. [To see this, note that we must have $P_j-\{w\} \neq \emptyset$ for otherwise G would be disconnected, contradicting our assumption. Since G is connected, it follows that $N(w) \subseteq N(a_j)$ and $V-(\{a_j\} \cup (P_j-\{w\}))$ is a star-cutset in $\bar{G}$, centered at w].

Step 4. If Q consists of sinks only, then let C be a component of P_i consisting of at least two vertices. (Such a component must exist or else G would be bipartite.)

4.1. If each $s \in Q$ is adjacent to either all the vertices of C or to none of them, then stop: for any $x \in C$ there is a star-cutset in $\bar{G}$ consisting of x together with all vertices in $V-C$ that are not adjacent to x in G.

4.2. Now we may assume that some $s \in Q$ is adjacent to some, but not all, vertices in C. Since C is connected in G, it follows that there are adjacent vertices x, y in C with $x \to s$, $ys \notin E$. Since we did not stop in Step 3, there exists a vertex $t \in Q$ with $y \to t$. If $x \to t$ then stop: Lemma 1 with all the directions reversed and with $a = s$, $c = t$ guarantees that $\bar{G}$ has a star-cutset.

Now $xt \notin E$. Note that we must have $N(s) \subseteq C$ or $N(t) \subseteq C$, for otherwise there would be vertices u, v in $P - C$ with $u \to s$, $v \to t$ and so $usxy$ or $vtyx$ would be a bad P_4. But now $V - (\{a_i\} \cup P_i)$ is a star-cutset in $\bar{G}$, centered at r for $r = s$ or $r = t$.

Step 5. Now there exist mixed vertices in Q. Let z stand for the first mixed vertex in Q in the linear order $<$. Write $Z_1 = N(z) \cap Q$ and $Z_2 = N(z) \cap P$.

By minimality of z we have $Z_2 \neq \emptyset$.

5.1. If $Z_1 = \emptyset$ then $N(z) \subseteq N(a_j)$ for some j, for otherwise since z is a mixed vertex we find vertices $u \in P_i$, $v \in P_j$ such that $z \to u$ and $v \to z$, implying that $a_j vzu$ is a bad P_4, a contradiction. Clearly $V - (\{a_j\} \cup P_j)$ is a star-cutset in $\bar{G}$, centered at z.

5.2. Now $Z_1 \neq \emptyset$. We claim that every vertex $u \in Z_1$ is adjacent to every vertex $v \in Z_2$, for otherwise $uzva_j$, with $a_j \to v$, would be a bad P_4. But now $\bar{G}$ has a star-cutset consisting of $V - N(z)$ and centered at z. ∎

LEMMA 2. *Every trace of a cycle of length at least five meets all of the following sets:*

$$W_1 = LMN \qquad W_4 = CEFK \qquad W_7 = ABHL$$
$$W_2 = GHK \qquad W_5 = CMN \qquad W_8 = ABLM$$
$$W_3 = ABDL \qquad W_6 = CFGK \qquad W_9 = CEKN.$$

Proof of Lemma 2. Let T be the trace of a cycle of length at least five. We may assume that the vertices of the cycle are 1, 2, ..., n.

1. $T \cap W_1 \neq \emptyset$. Let i, j $(i<j)$ be the two neighbours of 1 and let k be the neighbour of i distinct from 1. If $1<k<i$ then $L \in T$; if $i<k<j$ then $N \in T$; if $j<k$ then $M \in T$.

2. $T \cap W_2 \neq \emptyset$. Let i, j $(i<j)$ be the two neighbours of 1 and let k be the neighbour of j distinct from 1. If $1<k<i$ then $H \in T$; if $i<k<j$ then $G \in T$; if $j<k$ then $K \in T$.

3. $T \cap W_3 \neq \emptyset$. Let i, j $(i<j)$ be the two neighbours of 1 and let k be the neighbour of i distinct from 1. If $1<k<i$ then $L \in T$. Thus we may

assume $i<k$. Now let t be the neighbour of k distinct from i. If $1<t<i$ then $D \in T$; if $i<t<k$ then $B \in T$; if $k<t$ then $A \in T$.

4. $T \cap W_4 \neq \varnothing$. Let i, j $(i<j)$ be the two neighbours of 1 and let k be the neighbour of j distinct from 1. If $j<k$ then $K \in T$. Thus we may assume $1<k<j$. Now let t be the neighbour of k distinct from j. If $1<t<k$ then $F \in T$; if $k<t<j$ then $E \in T$; if $j<t$ then $C \in T$.

5. $T \cap W_5 \neq \varnothing$. Let k be the smallest positive integer such that k is not a source; let i, j $(i<j)$ be the two neighbours of k. If k is a sink then let t be the neighbour of j distinct from k; by minimality of k, we have $k<t$ and so $C \in T$. If k is a mixed vertex then let t be the neighbour of i distinct from k. By minimality of k, we have $k<t$; if $k<t<j$ then $M \in T$; if $j<t$ then $N \in T$.

6. $T \cap W_6 \neq \varnothing$. Let k, i, j be as in the proof of $T \cap W_5 \neq \varnothing$; again, if k is a sink then $C \in T$. If k is mixed, then let x be the neighbour of i distinct from k. By minimality of k, we have $k<x$. Next, let y be the neighbour of x distinct from i. If $y<i$ then $E \in T$; if $k<y<x$ then $G \in T$; if $x<y$ then $K \in T$. Hence we may assume $i<y<k$. Finally, let z be the neighbour of y distinct from x. By minimality of k, we have $k<z$; if $k<z<x$ then $E \in T$; if $x<z$ then $C \in T$.

7. $T \cap W_7 \neq \varnothing$. Let k, i, j be as in the proof of $T \cap W_5 \neq \varnothing$. If k is a sink then let t be the neighbour of i distinct from k; by minimality of k, we have $k<t$, and so $L \in T$. If k is mixed then let x be the neighbour of j distinct from k. If $k<x<j$ then $B \in T$; if $j<x$ then $A \in T$. Hence we may assume $x<k$. Let y be the neighbour of x distinct from j. By minimality of k, we have $k<y$. If $k<y<j$ then $H \in T$; if $j<y$ then $L \in T$.

8. $T \cap W_8 \neq \varnothing$. Let k be the smallest positive integer such that k is a sink; let i, j $(i<j)$ be the two neighbours of k and let x be a neighbour of i distinct from k. If $x<i$ then $B \in T$; if $k<x$ then $L \in T$. Hence we may assume $i<x<k$. Let y be the neighbour of x distinct from i. By minimality of k, we have $x<y$; if $k<y$ the $M \in T$. Hence we may assume $x<y<k$. Finally, let z be the neighbour of y distinct from x. By minimality of k, we have $y<z$, and so $A \in T$.

9. $T \cap W_9 \neq \varnothing$. Let k, i, j be as in the proof of $T \cap W_8 \neq \varnothing$, and let x be the neighbour of j distinct from k. If $j<x<k$ then $E \in T$; if $k<x$ then $C \in T$. Hence we may assume $x<j$.

Set $v_0=k$, $v_1=j$, $v_2=x$, $t=2$ and, as long as v_t has a neighbour y with $y<v_t$, keep setting $v_{t+1}=y$, $t \leftarrow t+1$. When this process terminates v_t has a neighbour y with $y \neq v_{t-1}$, $v_t<y$. If $v_t<y<v_{t-1}$ then $K \in T$; if $v_{t-2}<y$ then $N \in T$. Hence we may assume that $v_{t-1}<y<v_{t-2}$; note that $v_{t-2}<k$.

Finally, let z be the neighbour of y distinct from v_t. By minimality of k, we have $y < z$, and so $K \in T$. ∎

For each ordered graph X with vertices 1, 2, ..., n we shall denote by

- $F(X)$ the ordered graph resulting from X when each vertex i is relabeled as $n+1-i$.
- $C(X)$ the ordered graph resulting from X when every two non-adjacent vertices are made adjacent and vice versa.

On the set of the 12 ordered graphs in Fig. 1 these two mappings act as shown in Fig. 2.

Note that, for every unordered graph Q and for each subset S of $\{A, B, ..., N\}$, the following three statements are equivalent:

(i) S meets the trace of every ordered graph isomorphic (as an unordered graph) to Q,

(ii) $F(S)$ meets the trace of every ordered graph isomorphic (as an unordered graph) to Q,

(iii) $C(S)$ meets the trace of every ordered graph isomorphic (as an unordered graph) to the complement $\bar{Q}$ of Q.

For further reference, observe that the family $S_1, S_2, ..., S_{42}$ is "closed" under applications of the mappings C and F: more precisely, for each S_i there are S_j, S_k such that $F(S_i) = S_j$ and $C(S_i) = S_k$ with $i, j, k = 1, 2, ..., 42$ (see Fig. 3).

A graph G is called *weakly triangulated* if it contains no induced subgraph isomorphic to a C_k or $\bar{C}_k$ with $k \geq 5$. Hayward [3] has proved that weakly triangulated graphs are perfect.

LEMMA 3. *Each of the sets S_i with $i = 1, 2, ..., 36$ has the following property: if the trace of an ordered graph Z is disjoint from S_i then Z is weakly triangulated.*

Proof of Lemma 3. Clearly, we only need prove that Z contains no induced subgraph isomorphic to a C_k or $\bar{C}_k$ with $k \geq 5$.

X	A	B	C	D	E	F	G	H	K	L	M	N
$F(X)$	A	K	C	M	L	N	G	H	B	E	D	F
$C(X)$	H	L	G	N	K	M	C	A	E	B	F	D

FIGURE 2

S_i	$F(S_i)$	$C(S_i)$	S_i	$F(S_i)$	$C(S_i)$
S_1	S_{12}	S_{23}	S_{22}	S_{11}	S_{36}
S_2	S_{13}	S_{24}	S_{23}	S_{30}	S_1
S_3	S_{14}	S_3	S_{24}	S_{31}	S_2
S_4	S_{15}	S_4	S_{25}	S_{32}	S_5
S_5	S_{16}	S_{25}	S_{26}	S_{33}	S_7
S_6	S_{17}	S_{17}	S_{27}	S_{34}	S_8
S_7	S_{18}	S_{26}	S_{28}	S_{35}	S_{10}
S_8	S_{19}	S_{27}	S_{29}	S_{36}	S_{11}
S_9	S_{20}	S_9	S_{30}	S_{23}	S_{12}
S_{10}	S_{21}	S_{28}	S_{31}	S_{24}	S_{13}
S_{11}	S_{22}	S_{29}	S_{32}	S_{25}	S_{16}
S_{12}	S_1	S_{30}	S_{33}	S_{26}	S_{18}
S_{13}	S_2	S_{31}	S_{34}	S_{27}	S_{19}
S_{14}	S_3	S_{14}	S_{35}	S_{28}	S_{21}
S_{15}	S_4	S_{15}	S_{36}	S_{29}	S_{22}
S_{16}	S_5	S_{32}	S_{37}	S_{38}	S_{39}
S_{17}	S_6	S_6	S_{38}	S_{37}	S_{40}
S_{18}	S_7	S_{33}	S_{39}	S_{40}	S_{37}
S_{19}	S_8	S_{34}	S_{40}	S_{39}	S_{38}
S_{20}	S_9	S_9	S_{41}	S_{41}	S_{42}
S_{21}	S_{10}	S_{35}	S_{42}	S_{42}	S_{41}

FIGURE 3

The cases of $i = 1, 2, \ldots, 11$ may be settled by observing that

$$
\begin{aligned}
S_1 &= W_4 \cup C(W_2) & S_7 &= F(W_3) \cup C(W_2)\\
S_2 &= W_5 \cup C(W_2) & S_8 &= F(W_3) \cup C(W_4)\\
S_3 &= W_6 \cup C(W_6) & S_9 &= F(W_5) \cup C(W_5)\\
S_4 &= W_7 \cup C(W_7) & S_{10} &= F(W_8) \cup C(W_2)\\
S_5 &= W_9 \cup C(W_2) & S_{11} &= F(W_8) \cup C(W_9).\\
S_6 &= F(W_2) \cup C(W_2)
\end{aligned}
$$

The cases of $i = 12, 13, \ldots, 22$ may be settled by observing that

$$
\begin{aligned}
S_{12} &= F(S_1) & S_{18} &= F(S_7)\\
S_{13} &= F(S_2) & S_{19} &= F(S_8)\\
S_{14} &= F(S_3) & S_{20} &= F(S_9)\\
S_{15} &= F(S_4) & S_{21} &= F(S_{10})\\
S_{16} &= F(S_5) & S_{22} &= F(S_{11}).\\
S_{17} &= F(S_6)
\end{aligned}
$$

Finally, the cases of $i = 23, 24, \ldots, 36$ may be settled by observing that

$$\begin{array}{ll} S_{23} = C(S_1) & S_{30} = C(S_{12}) \\ S_{24} = C(S_2) & S_{31} = C(S_{13}) \\ S_{25} = C(S_5) & S_{32} = C(S_{16}) \\ S_{26} = C(S_7) & S_{33} = C(S_{18}) \\ S_{27} = C(S_8) & S_{34} = C(S_{19}) \\ S_{28} = C(S_{10}) & S_{35} = C(S_{21}) \\ S_{29} = C(S_{11}) & S_{36} = C(S_{22}). \end{array}$$

Proof of Theorem 1. Note that for sets S_i with $i = 1, 2, \ldots, 36$ the desired conclusion follows immediately from Lemma 3 and Hayward's result.

Next, for sets S_i with $i = 37, 38, 39, 40$ the conclusion follows as a corollary to Chvátal's theorem.

Finally, to complete the proof we shall rely on the following corollary of Theorem 2:

Opposion graphs are perfect.

[Assuming the contrary, a minimal counterexample would be minimal imperfect. By Theorem 2 and the star-cutset lemma it follows that $\bar{G}$ is perfect. Now by a fundamental result of Lovász [4], G is perfect if and only if $\bar{G}$ is perfect, a contradiction.]

Now we only need observe that if the trace of an ordered graph Z is disjoint from S_{42}, then Z is an opposition graph, and that $C(S_{42}) = S_{41}$. This settles the cases of S_i with $i = 41, 42$. ∎

We shall say that a graph G is *breakable* (a term coined by V. Chvátal) if at least one of G, $\bar{G}$ has a star-cutset.

Chvátal [2] proved that if a graph G is perfectly orderable, then it is either bipartite or else $\bar{G}$ has a star-cutset.

Hayward [3] proved that if a graph is weakly triangulated then it is breakable.

These facts combined with Theorem 2 yield the following proposition:

If the trace of an ordered graph Z is disjoint from some S_i with $i = 1, 2, \ldots, 42$ then Z is breakable or else Z or $\bar{Z}$ is bipartite.

By virtue of the Star-Cutset Lemma, this proposition implies Theorem 1.

We note that the converse of Theorem 1 is false: there exist graphs which are perfect but have no ordering which is one of these variations on perfectly orderable graphs. One such graph H is featured in Fig. 4.

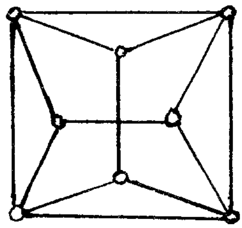

FIGURE 4

To justify this claim it suffices to note that the graph H is perfect, neither H nor $\bar{H}$ is bipartite and that H is not breakable.

To prove the completeness of the list $S_1, S_2, \ldots, S_{42}$ we prove that every set that could play the role of some S_i with $i = 1, 2, \ldots, 42$ in Theorem 1 must contain some S_j with $j = 1, 2, \ldots, 42$. The proof itself is long and tedious but the basic idea is quite simple: we identify a family of sets $T_j, j = 1, 2, \ldots, 118$ such that each T_j is either the trace of an odd cycle or the trace of the complement of an odd cycle. We then prove that if a subset S of $AB \ldots N$ meets each T_j with $j = 1, 2, \ldots, 118$, then S contains some S_i with $i = 1, 2, \ldots, 42$.

The interested reader can find the details in [5].

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